# A universal property of the Drinfeld double of a finite dimensional Hopf algebra

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*Dedicated to the dead in the landslide accident of Zhenxiong, Yunnan Province on Janurary 22nd, 2024 in China.*

Universal properties are rudimentary in mathematics. A universal proprety characterizes some object $X$ via relations with all objects that have more or less the same flavour as $X$. For example, the universal enveloping algebra of a Lie algebra $g$ is the unique associative algebra that classifies all Lie algebra homomorphisms from $g$ to algebras.

On the other hand, Hopf algebras play a pivotal role in modern mathematics, partly due to their connection with manifold topology, combinatorics and other fields, and are interesting subjects in their own rights. They also have great impact on physics via conformal field theory, topological orders etc.

In this talk, I will present a universal property of a well-knowned construction, the Drinfeld double $D(H)$ of a finite dimensional Hopf algebra $H$. The universal property is not surprising, but seems to be missing in the literature.

The universal property of $D(H)$ are the same as, in certain sense, that satisfied by the **center** of a $\Bbbk$-algebra. Thus I will divide my talk into two parts:

(A) First, I focus on centers.

(B) Secondly, I formulate the result. Then instead of proving it, I show the "unsurprising-ness" of the result, i.e., why one can expect such a result is true.

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The work is to appear on arXiv.

# Centers

We fix a field $\mathbb{k}$. Let $A$ be a $\mathbb{k}$-algebra. The center of $A$ is defined to be the subalgebra

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It is observed, for e.g. by Lurie, that the center can be completely characterized by that

- $Z(A)$ is a $\mathbb{k}$-algebra, equipped with an algebra homomorphism $\rho\colon Z(A) \otimes_{\mathbb{k}} A \to A$ such that $\rho(1_Z \otimes -) = \mathrm{id}$, where $1_Z$ is the unit of $Z(A)$;
- It is universal: given a $\mathbb{k}$-algebra $B$ together with an algebra homomorphism $\lambda\colon B \otimes_{\mathbb{k}} A \to A$ such that $\lambda(1_B \otimes -) = \mathrm{id}$, there exists uniquely an algebra homomorphism $\underline{\lambda}\colon B \to Z(A)$ such that $\lambda = \rho \circ (\underline{\lambda} \otimes \mathrm{id})$, i.e., all actions on $A$ factor through $Z(A) \otimes A \to A$ uniquely.

$$B \otimes A \xrightarrow[\underline{\lambda} \otimes \mathrm{id}_A]{} Z(A) \otimes A \xrightarrow[\rho]{} A, \qquad \lambda\colon B \otimes A \to A$$

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**Remark.** The condition "universal" guarantees that if the center of $A$ exists, it is uniquely determined.

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Check:

➡ Define $\rho\colon Z(A) \otimes_{\Bbbk} A \to A, z \otimes a \mapsto z \cdot a$. Then $\rho$ is an algebra homomorphism since $\rho(z_1 \otimes a_1)\rho(z_2 \otimes a_2) = z_1 a_1 z_2 a_2 = z_1 z_2 a_1 a_2 = \rho(z_1 z_2 \otimes a_1 a_2)$; moreover, we have $\rho(1 \otimes -) = \mathrm{id}\colon A \to A$.

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Check (cont'd):

➡ (Universal) Suppose $\lambda\colon B \otimes_{\Bbbk} A \to A$ is an algebra homomorphism, then $\underline{\lambda}\colon B \to Z(A)\colon b \mapsto \lambda(b \otimes 1_A)$ is indeed well-defined as $a\lambda(b \otimes 1_A) = \lambda(b \otimes a) = \lambda(b \otimes 1_A)a$ for all $a \in A$. Moreover, $\underline{\lambda}$ is an algebra homomorphism.

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Check (cont'd):

➡ (Universal cont'd) We also have $(\rho \circ (\underline{\lambda} \otimes \mathrm{id}))(b \otimes a) = \lambda(b \otimes 1_A)a = \lambda(b \otimes 1_A)\lambda(1_B \otimes a) = \lambda(b \otimes a)$ for all $b \in B, a \in A$, hence $\rho \circ (\underline{\lambda} \otimes \mathrm{id}) = \lambda\colon B \otimes_{\Bbbk} A \to A$. Moreover, the algebra homomorphism $\underline{\lambda}$ satisfying this condition is unique. ✓

# Formalism

We can formalize the story in previous slides using the notion of a *monoidal category*:

$$\begin{aligned}
\mathbb{k}\text{-algebras} &\rightsquigarrow \text{objects in a monoidal category } \mathcal{C} \\
\mathbb{k}\text{-algebra homomorphism} &\rightsquigarrow \text{morphisms in } \mathcal{C} \\
\otimes_{\mathbb{k}} &\rightsquigarrow \text{the tensor product } \otimes \text{ in } \mathcal{C} \\
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**Definition**

A **monoidal category** is a category $\mathcal{C}$ with functors $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\hat{I} \colon \{*\} \to \mathcal{C}, * \mapsto I$ equipped coherently with natural isomorphisms

- $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
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(2)Let $(C, \cdot, 1_C)$ be a monoid. Then $C$ can be viewed as a monoidal category with only trivial morphisms, where the tensor product is given by $X \otimes Y := X \cdot Y$.
(3)The category $\mathrm{LMod}(H)$ of finite dimensional left modules over a Hopf algebra $H$ is a monoidal category. In particular, the category $\mathrm{Rep}(G)$ of representations over a finite group $G$ is a monoidal category. These monoidal categories are built from the monoidal category $(\mathrm{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ of $\Bbbk$-vector spaces with tensor product being relative tensor products over $\Bbbk$.

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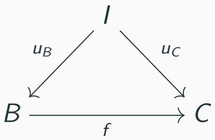
# Center: definition (Preparation)

**Definition.** Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. An $E_0$-**algebra** in $\mathcal{C}$ is a pair $(B, u_B\colon I \to B)$ where $B \in \mathcal{C}$ and $u_B \in \mathrm{Mor}(\mathcal{C})$. An $E_0$-**algebra homomorphism** $(B, u_B) \to (C, u_C)$ is a morphism $f\colon B \to C \in \mathrm{Mor}(\mathcal{C})$ such that the diagram at left commutes.

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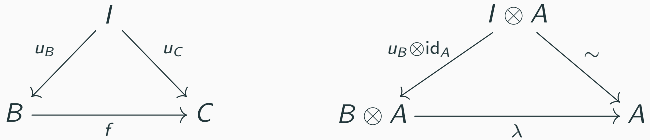


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**Definition.** Let $(B, u_B)$ be an $E_0$-algebra in $\mathcal{C}$ and $A \in \mathcal{C}$ be an object. A **left unital action** of $(B, u_B)$ on $A$ is a morphism $\lambda\colon B \otimes A \to A$ such that the diagram at right commutes.

The set of left unital actions of $B$ on $A$ is denoted by $\mathrm{LUA}_A(B)$.

# Center: definition

**Definition.** Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and $A \in \mathcal{C}$. The **center** of $A$ is an $E_0$-algebra $(Z(A) \in \mathcal{C}, u_Z : I \to Z(A))$ in $\mathcal{C}$ equipped with a left unital action $\rho_Z : Z(A) \otimes A \to A$ on $A$, such that:

- Given any $E_0$-algebra $(B, u_B)$ and a left unital action $\lambda \colon B \otimes A \to A$, there exists a unique $E_0$-algebra homomorphism $\underline{\lambda} \colon B \to Z(A)$ such that $\lambda = \rho_Z \circ (\underline{\lambda} \otimes \mathrm{id})$, i.e., the diagram

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Equivalently, one defines the center $(Z(A), \rho_Z)$ to be the unique left unital action such that for all $E_0$-algebras $B$, the pushforward

$$E_0(\mathcal{C})(B, Z(A)) \to \mathrm{LUA}_A(B), f \mapsto \rho_Z \circ (f \otimes \mathrm{id})$$

is an isomorphism, where the uniqueness follows from Yoneda lemma in disguise.

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$$\overbrace{B \otimes A \xrightarrow[\underline{\lambda} \otimes \mathrm{id}_A]{} Z(A) \otimes A \xrightarrow[\rho_Z]{} A}^{\lambda} \quad \text{commutes.}$$

**Example.** Let $\mathcal{C} = \mathrm{Alg}(\mathrm{Vec}_{\Bbbk})$. Then the center of $A \in \mathcal{C}$ recovers the usual center of $A$, equippd with the map $\rho \colon Z(A) \otimes A \to A, z \otimes a \mapsto za$.

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$$B \otimes A \xrightarrow{\underline{\lambda} \otimes \mathrm{id}_A} Z(A) \otimes A \xrightarrow{\rho_Z} A \quad (\text{with } \lambda \colon B \otimes A \to A) \quad \text{commutes.}$$

**Example.** Let $\mathcal{C} = \mathrm{Vec}_{\Bbbk}$. Then the center of $V \in \mathcal{C}$ is the space $\mathrm{End}(V)$ equipped with the canonical evaluation map $\mathrm{ev} \colon \mathrm{End}(V) \otimes_{\Bbbk} V \to V, f \otimes v \mapsto f(v)$.

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**Example.** More generally, in a monoidal category $\mathcal{C}$ with internal homs, the center of $x \in \mathcal{C}$ is given by the internal hom $[x, x]$. For example, when $\mathcal{C}$ is a fusion category, there is $Z(x) = x \otimes x^L$.

# Center: definition (cont'd)

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**Example.** If the monoidal category is coCartesian, i.e., the monoidal structure is given by coproduct, then $Z(A) = A$. In particular, the center of a commutative algebra is itself.

Note that in a monoidal category, algebras along with their left modules can be defined, directly generalizing their $\mathbb{k}$-linear counterparts. Then the following observation is clear:

**Proposition**

(1) The center $Z(A)$ is an algebra in $\mathcal{C}$, and the action $\rho_Z \colon Z(A) \otimes A \to A$ is a left module action on $A$.

(2) The center is terminal in all such left module actions on $A$. That is, suppose $B$ is an algebra in $\mathcal{C}$ and $\lambda \colon B \otimes A \to A$ is a left module action on $A$, then the unique morphism $\underline{\lambda} \colon B \to Z(A)$ is an algebra homomorphism.

**Proof.**

(1). The multiplication $m_Z \colon Z(A) \otimes Z(A) \to Z(A)$ is induced from the left unital action $Z(A) \otimes Z(A) \otimes A \xrightarrow{\mathrm{id} \otimes \rho_Z} Z(A) \otimes A \xrightarrow{\rho_Z} A$. The defining property of $m_Z$ reads $\rho_Z \circ (m_Z \circ \mathrm{id}) = \rho_Z \circ (\mathrm{id} \otimes \rho_Z)$, which precisely dicates that $A$ is a left $Z(A)$-module.

(2). Similiarly as (1), using the universal property of $Z(A)$. □

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**Example.** $Z(A) \in \mathsf{Alg}(\mathsf{Alg}(\mathrm{Vec}_{\Bbbk}))$ for $A \in \mathsf{Alg}(\mathrm{Vec}_{\Bbbk})$. By Eckman-Hilton argument, an **algebra in the monoidal category of algebras** in $\mathrm{Vec}_{\Bbbk}$ is precisely a commutative algebra. So the universal property of center presents us with a fancy way of stating the simple fact that the center of an algebra is a commutative algebra.

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2-Functors and higher natural transformations are easy to define.

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- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
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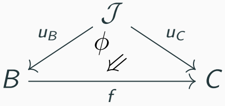
**Definition.** An $E_0$-**algebra** in a monoidal 2-category $\mathcal{C}$ is a pair $(B, u_B \colon \mathcal{J} \to B)$ where $B \in \mathcal{C}$ and $u_B \in 1\,\mathrm{Mor}(\mathcal{C})$. An $E_0$-**algebra homomorphism** $(B, u_B) \to (C, u_C)$ is a pair $(f, \phi)$ where $f \in 1\,\mathrm{Mor}(\mathcal{C})$ and $\phi$ is an invertible 2-morphism displayed in the diagram at the left.



A monoidal 2-category is a 2-category $\mathcal{C}$ equipped with 2-functors $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\hat{\mathcal{J}}\colon \{*\} \to \mathcal{C}, * \mapsto \mathcal{J}$ equipped with the three adjoint 2-natural equivalences

- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- $X \otimes \mathcal{J} \xrightarrow{\sim} X \simeq \mathcal{J} \otimes X$ for all $X \in \mathcal{C}$,

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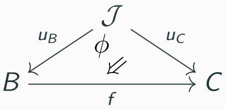
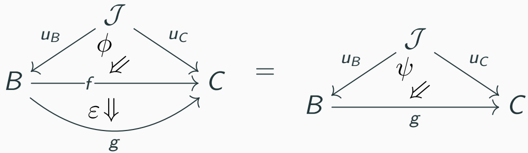
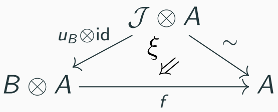



An $E_0$-**algebra 2-homomorphism** $(f, \phi) \Rrightarrow (g, \psi)$ is a 2-morphism $\varepsilon\colon f \Rightarrow g$ such that the equation of pasting diagram at the right holds.

We denote by $\mathcal{E}_0(\mathcal{C})$ the 2-category of $E_0$-algebras in $\mathcal{C}$.

**Definition.** Let $(B, u_B) \in \mathcal{E}_0(\mathcal{C})$ and $A \in \mathcal{C}$ be an object. A **left unital action of** $B$ **on** $A$ is a pair $(f : B \otimes A \to A, \xi)$ where $f \in 1\,\mathrm{Mor}(\mathcal{C})$, and $\xi$ is an invertible 2-morphism witnessing the "left unitality" of the morphism $f$ (called **left unitality structure**), as displayed below.



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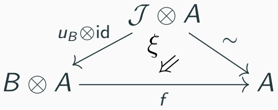
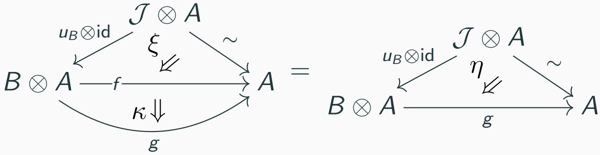



A **homomorphism of left unital actions** $(f, \xi) \Rightarrow (g, \eta)$ is a 2-morphism $\kappa\colon f \Rightarrow g$ such that the equation of pasting diagrams in the diagram holds.

We denote by $\mathcal{LUA}_A(B)$ the category of left unital actions.

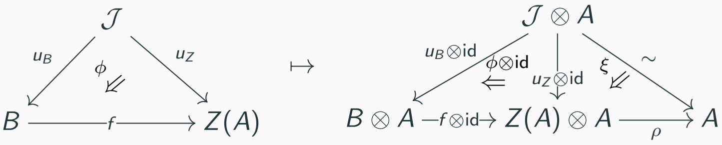
**Definition.**

The **center** $Z(A)$ of $A \in \mathcal{C}$ is an $E_0$-algebra in $\mathcal{C}$ together with a left unital action $(\rho\colon Z(A) \otimes A \to A, \xi)$ such that the pushforward functor

$$\mathcal{E}_0(\mathcal{C})(B, Z(A)) \to \mathcal{LUA}_A(B)$$

by $(\rho, \xi)$ is an equivalence for all $B \in \mathcal{E}_0(\mathcal{C})$.

The action of the pushforward functor is illustrated as below:



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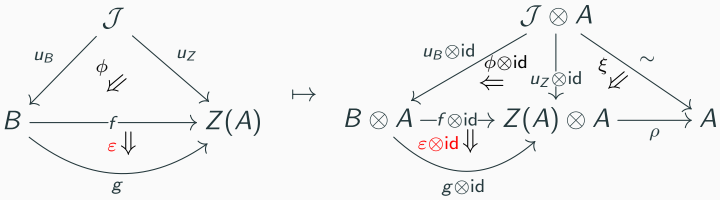

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The take-away: The center still carries an action $\rho \colon Z(A) \otimes A \to A$, this time being left unital up to an extra data.

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**Examples (Without proof).** (1) In the monoidal 2-category $(\mathrm{Cat}, \times, \{*\})$ of categories, functors and natural transformations, the center of a category $\mathcal{A} \in \mathrm{Cat}$ is the category $\mathrm{End}(\mathcal{A})$ of functors $\mathcal{A} \to \mathcal{A}$ equipped with the canonical evaluation functor $\mathrm{ev}\colon \mathrm{End}(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}$.

# 2-categorical center (cont'd)

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**Examples (Without proof).** (1) In the monoidal 2-category $(\mathrm{Cat}, \times, \{*\})$ of categories, functors and natural transformations, the center of a category $\mathcal{A} \in \mathrm{Cat}$ is the category $\mathsf{End}(\mathcal{A})$ of functors $\mathcal{A} \to \mathcal{A}$ equipped with the canonical evaluation functor $\mathsf{ev}\colon \mathsf{End}(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}$.
(2) In the 2-category $\mathrm{MonCat} = \mathsf{Alg}(\mathrm{Cat})$ of monoidal categories, monoidal functors and monoidal natural transformations, the center $Z(\mathcal{A})$ of $(\mathcal{A}, \otimes, I) \in \mathrm{MonCat}$ is called the **Drinfeld center**, and is a monoidal category whose set of objects are

$$Z(\mathcal{A}) := \{(W \in \mathcal{A}, \gamma_{-,W})\},$$

where $\gamma_{-,W}$ is a half-braiding, i.e., a family $\{\gamma_{X,W}\colon X \otimes W \xrightarrow{\sim} W \otimes X\}_{X \in \mathcal{A}}$ of isomorphisms natural in $X$ and satisfying certain coherence relations.

# Example: Drinfeld center

**Examples.** (2)(cont'd) The universal action reads

$$\rho\colon Z(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}, ((W, \gamma_{-,W}), A) \mapsto W \otimes A.$$

It is a monoidal functor with the monoidal structure given by the half-braiding $\gamma$'s. Namely, we use the half-braiding $\gamma_{A_1,W_2}$ to build the isomorphism $T\colon W_1 \otimes A_1 \otimes W_2 \otimes A_2 \xrightarrow{\sim} W_1 \otimes W_2 \otimes A_1 \otimes A_2$ for $W_1, W_2 \in Z(\mathcal{A})$ and $A_1, A_2 \in \mathcal{A}$. The coherence conditions satisfied by the half-braiding precisely makes $T$ a well-defined monoidal structure. To completely check that $\rho$ indeed satisfies the universal property is rather tedious though straightforward; see also Example 5.3.1.18 in Lurie's *Higher Algebra*.

# Example: Drinfeld center (cont'd)

I would like to give some examples of Drinfeld centers.

**Examples of Drinfeld centers.** (1) Suppose $C$ is the monoidal category obtained from a monoid $(C, \cdot, 1_C)$ by adding trivial morphisms. Then the Drinfeld center of $C$ is the center of the monoid $C$ (with trivial morphisms.)
(2) Let $G$ be a finite group. Then the Drinfeld center $Z(\text{Rep}(G))$ of the representation category of $G$ is the category of $G$-graded $G$-representations in which the grading respects the $G$-action in a certain way.
(3) Let $H$ be a finite dimensional Hopf algebra. Then the Drinfeld center $Z(\mathrm{LMod}(H))$ is the representation category of the Drinfeld double $D(H)$ of $H$.

**Remark.** In a monoidal 2-category $\mathcal{C}$, there is a notion of algebras (= pseudomonoid in the sense of [Day and Street: 1997]), and it is easy to see that the center

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(1) The case $\mathcal{C} = \mathrm{Cat}$. Note that algebras in $\mathrm{Cat}$ are precisely monoidal categoires, i.e., there is $\mathsf{Alg}(\mathrm{Cat}) = \mathrm{MonCat}$; and indeed we have $\mathsf{End}(\mathcal{A}) \in \mathsf{Alg}(\mathrm{Cat}) = \mathrm{MonCat}$ for $\mathcal{A} \in \mathcal{C}$.

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(2) The case $\mathcal{C} = \mathrm{MonCat}$. Note that [Joyal and Street: 1993] has shown that there is a canonical 2-equivalence $\mathsf{Alg}(\mathsf{Alg}(\mathrm{Cat})) = \mathrm{BrMonCat}$ of 2-categories, where $\mathrm{BrMonCat}$ is the 2-category of braided monoidal categories. Hence the Drinfeld double $Z(\mathcal{A})$ has a canonical structure of a braided monoidal category for $\mathcal{A} \in \mathrm{MonCat}$.

# Short summary

We have seen:

- Center of an object in a monoidal category. For e.g., center of a $\mathbb{k}$-algebra, center of a vector space.
- Center of an object in a monoidal 2-category. For e.g., center of a monoidal category.

# The universal property of Drinfeld double

My aim is to state the result and show why we can expect such a result is true. The rest of my talk is mostly devoted to make the necessary prepration, which is also the key ( and the only) ingredient of the proof: Tannak-Krein duality.

- Preparation: Tannaka-Krein duality and quasi-bialgebras.
- Stating the main result and show how it can be expected.

# Tannaka-Krein duality

Tannaka-Krein duality deals with the relation between algebras and its category of left modules/representations. From now on we assume finite dimensionality for both algebras and their modules.

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$$A \xrightarrow{\phi} B \ : \ \text{algebra homomorphism} \quad \overset{\mathrm{TK}}{\longmapsto} \quad \mathrm{LMod}(A) \xleftarrow{\phi^*} \mathrm{LMod}(B) \ : \ \text{functor}$$

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$$A \underset{\psi}{\overset{\phi}{\rightrightarrows}} B \;\; (b \Downarrow) \; : \; \text{“intertwinner”} \qquad \overset{\text{TK}}{\longmapsto} \qquad \mathrm{LMod}(A) \underset{\psi^*}{\overset{\phi^*}{\leftleftarrows}} \mathrm{LMod}(B) \;\; (b^* \Downarrow) \; : \; \text{natural transformation}$$

Here an “intertwinner” $b \colon \phi \Rightarrow \psi$ between algebra homomorphisms is an element $b \in B$ such that $b\phi(a) = \psi(a)b$ for all $a \in A$. Then the component of the natural transformation $b^*$ reads $b.- \colon \phi^*({}_B V) \to \psi^*({}_B V)$ for ${}_B V \in \mathrm{LMod}(B)$.

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**Remark.** Note that on this last level the action of $\mathrm{TK}$ is *fully-faithful*, in the sense that all natural transformations $\alpha\colon \phi^* \Rightarrow \psi^*$ arise in this way for a unique $b \in B$.

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**Remark (cont’d).** However, on the 1-morphisms’ level, the action of $\mathrm{TK}$ is far from surjective.

# Exercise for TK-duality: Equip $\mathrm{LMod}(H)$ with a monoidal structure!

Exercise! Suppose $H$ is an algebra. How to equip $\mathrm{LMod}(H)$ with a monoidal structure assuming you know TK duality?

Functor $\otimes\colon \mathrm{LMod}(H)\times\mathrm{LMod}(H)\to\mathrm{LMod}(H\otimes_{\mathbb{k}} H)\to\mathrm{LMod}(H)$

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Natural isomorphism

$$\begin{array}{ccc} \mathrm{LMod}(H)\times\mathrm{LMod}(H)\times\mathrm{LMod}(H) & \xrightarrow{\otimes\times\mathrm{id}} & \mathrm{LMod}(H)\times\mathrm{LMod}(H) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\otimes} & \alpha \Swarrow & \Big\downarrow{\scriptstyle \otimes} \\ \mathrm{LMod}(H)\times\mathrm{LMod}(H) & \xrightarrow[\otimes]{} & \mathrm{LMod}(H) \end{array}$$

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$$\text{Invertible intertwinner} \quad \longmapsto \quad \text{Natural isomorphism}$$

$$\begin{array}{ccc} H\otimes_{\Bbbk}H\otimes_{\Bbbk}H & \xleftarrow{\Delta\otimes\mathrm{id}} & H\otimes_{\Bbbk}H \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & a \Swarrow & \uparrow{\scriptstyle\Delta} \\ H\otimes_{\Bbbk}H & \xleftarrow[\Delta]{} & H \end{array} \qquad \begin{array}{ccc} \mathrm{LMod}(H)\times\mathrm{LMod}(H)\times\mathrm{LMod}(H) & \xrightarrow{\otimes\times\mathrm{id}} & \mathrm{LMod}(H)\times\mathrm{LMod}(H) \\ {\scriptstyle \mathrm{id}\times\otimes}\downarrow & \alpha \Swarrow & \downarrow{\scriptstyle\otimes} \\ \mathrm{LMod}(H)\times\mathrm{LMod}(H) & \xrightarrow[\otimes]{} & \mathrm{LMod}(H) \end{array}$$

## Hopf algebras

Summarising the structures we get on $H$, we have:

**Definition.** A **quasi-bialgebra** is an algebra $H$ equipped with algebra homomorphisms $\Delta\colon H \to H \otimes H, \varepsilon\colon H \to \Bbbk$, and an invertible interwinner $a \in H \otimes H \otimes H$ in the sense that

$$a \cdot (\Delta \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \Delta)\Delta(h) \cdot a, \forall h \in H,$$

subject to conditions

$$(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta\colon H \to H$$
$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(a) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(a) = (1_H \otimes a) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(a) \cdot (a \otimes 1_H)$$
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**Proposition.** If $H$ is a quasi-bialgebra, then $\mathrm{LMod}(H)$ is a monoidal category.

**Definition.** A **bialgebra** is a quasi-bialgebra $H$ with $a = 1_H \otimes 1_H \otimes 1_H$. In particular, in a bialgebra, $(H, \Delta, \varepsilon)$ always form a coassociative coalgebra. A **Hopf algebra** is a bialgebra satisfying the **property** of admitting an "antipode".

Let $H, K$ be quasi-bialgebras.

Functor $F\colon \mathrm{LMod}(H) \to \mathrm{LMod}(K)$

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A monoidal structure on $F$:

$$\begin{array}{ccc} \mathrm{LMod}(H)\times\mathrm{LMod}(H) & \longrightarrow & \mathrm{LMod}(H) \\ {\scriptstyle F\times F}\big\downarrow & \nearrow\!\!\!\nearrow\ T_2 & \big\downarrow{\scriptstyle F} \\ \mathrm{LMod}(K)\times\mathrm{LMod}(K) & \longrightarrow & \mathrm{LMod}(K) \end{array}$$

and

$$\begin{array}{ccc} \{*\} & \longrightarrow & \mathrm{LMod}(H) \\ & \searrow \quad \nearrow\!\!\!\nearrow\ T_0 & \big\downarrow{\scriptstyle F} \\ & & \mathrm{LMod}(K) \end{array}$$

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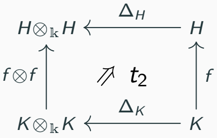
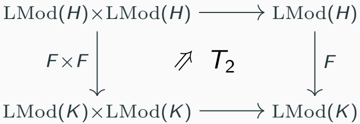
Interwinners $\quad \longmapsto \quad$ A monoidal structure on $F$:

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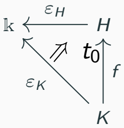


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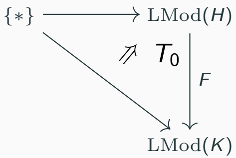


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subject to conditions

$$a_H \cdot (\Delta_H \otimes \mathrm{id})(t_2) \cdot (t_2 \otimes 1_H) = (\mathrm{id} \otimes \Delta_H)(t_2) \cdot (1_H \otimes t_2) \cdot f^{\otimes 3}(a_K)$$
$$t_0(\varepsilon_H \otimes \mathrm{id})(t_2) = 1_H = t_0(\mathrm{id} \otimes \varepsilon_H)(t_2).$$

**Definition.** Let $H, K$ be quasi-bialgebras. A **quasi-bialgebra homomorphism** $K \to H$ is an algebra homomorphism $f : K \to H$ equipped with invertible intertwinners $t_2 \in H \otimes_{\mathbb{k}} H$ and $t_0 \in \mathbb{k}$ in the sense that

$$t_2 \cdot (f \otimes f)\Delta_K(k) = \Delta_H(f(k)) \cdot t_2, \forall k \in K$$
$$t_0 \cdot \varepsilon_K(k) = \varepsilon_H(f(k)) \cdot t_0, \forall k \in K,$$

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**Remark.** If we take $f$ to be an isomorphism (or WLOG, the identity map), then a quasi-bialgebra homomorphism $(f, t_2, 1)$ recovers the notion of *twisting* by $t_2$.

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**Definition.** Let $(f, t_2, t_0), (g, s_2, s_0)\colon K \to H$ be quasi-bialgebra homomorphisms. A **quasi-bialgebra 2-homomorphism** $(f, t) \Rightarrow (g, s)$ is an intertwinner $\eta\colon f \Rightarrow g \in H$ such that

$$\Delta_H(\eta) \cdot t_2 = s_2 \cdot (\eta \otimes \eta), \quad \varepsilon_H(\eta)t_0 = s_0.$$

Then we have:

$$\text{quasi-bialgebra homomorphisms} \overset{\mathrm{TK}}{\longmapsto} \text{monoidal functors}$$

$$\text{quasi-bialgebra 2-homomorphisms} \overset{\mathrm{TK}}{\longmapsto} \text{monoidal natural transformations.}$$

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**Fact (McCrudden)**

Quasi-bialgebras, quasi-bialgebra homomorphisms, quasi-bialgebra 2-homomorphisms form a 2-category; it is a monoidal 2-category under $(\otimes_{\mathbb{k}}, \mathbb{k})$.

**Remark.** A more systematic way of understanding $\mathrm{QB}$ is as follows [Day, McCrudden, Street]: we have a **symmetric** monoidal 2-category $\mathcal{A}\mathrm{lg}_{\Bbbk}^{1-\mathrm{op}}$ of $\Bbbk$-algebras, reverse algebra homomorphisms and intertwinners. Then we have $\mathrm{QB}^{1-\mathrm{op}} = \mathsf{Alg}(\mathcal{A}\mathrm{lg}_{\Bbbk})$, i.e., the 2-category of quasi-bialgebras are the 2-category of algebras in $\mathcal{A}\mathrm{lg}_{\Bbbk}^{1-\mathrm{op}}$. Now the symmetric (lax) monoidal 2-functor

$$\mathrm{TK}\colon \mathcal{A}\mathrm{lg}_{\Bbbk}^{1-\mathrm{op}} \to \mathrm{Cat}$$

preserve algebras, thus the $\mathrm{LMod}(B) \in \mathsf{Alg}(\mathrm{Cat}) = \mathrm{MonCat}$, i.e., is a monoidal category.

Quasi-triangular quasi-bialgebras/triangular quasi-bialgebras could be understood in a similiar way, as they are precisely *braided algebras/symmetric algebras* in $\mathcal{A}\mathrm{lg}_{\Bbbk}^{1-\mathrm{op}}$, and are preserved by symmetric monoidal 2-funcotrs. Since we do not talk much about quasi-triangular and triangular quasi-bialgebras, we did not emphasize this viewpoint.

# Main Theorem (Preparation)

Now we introduce our main theorem, beginning with the Drinfeld double. Let $H$ be a finite dimensional Hopf algebra.

- Its antipode $S\colon H \to H$ is an anti-isomorphism and also a coalgebra-isomorphism satisfying certain relations.
- Note that the dual space $H^* := \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ has a natural Hopf algebra structure; we use $(H^{\mathrm{op}})$ to refer to the same Hopf algebra $H$ with multiplication reversed.

The **Drinfeld double** $D(H)$ ([Drinfeld: 1987]) of $H$ is the bialgebra whose underlying coalgebra is

$$(H^{\mathrm{op}})^* \otimes_{\Bbbk} H,$$

and whose multiplication is given by

$$(f \otimes a) \cdot (g \otimes b) := f \cdot g(S^{-1}(a_{(3)}) - a_{(1)}) \otimes a_{(2)} b.$$

**Fact** [Drinfeld, Majid, Kassel]

$D(H)$ is a Hopf algebra, and there is a canonical equivalence $J\colon \mathrm{LMod}(D(H)) \cong Z(\mathrm{LMod}(H))$.

**Sketch proof.**

Let ${}_{D(H)}V$ be a left $D(H)$-module. Then the underlying object of $J(V)$ is ${}_{H \hookrightarrow H^{\mathrm{op}*} \otimes H}V \in \mathrm{LMod}(H)$, while the half-braiding is given by

$$\gamma_{W,V}\colon W \otimes V \to V \otimes W, w \otimes v \mapsto e^i.v \otimes e_i.w,$$

for $W \in \mathrm{LMod}(H)$, where $\{e_i\}_i$ is a basis of $H$ and $e^i$ its dual.

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Conversely, let $(V, \gamma_{-,V} = \{\gamma_{U,V}\colon U \otimes V \xrightarrow{\sim} V \otimes U\}) \in Z(\mathrm{LMod}(H))$. Then we define $J^{-1}((V,\gamma))$ to be the vector space $V$ equipped with a $D(H)$-action $(\omega \otimes a).v := (\mathrm{id} \otimes \omega)(\gamma_{H,V}(1_H \otimes a.v))$ for $v \in V, \omega \otimes a \in D(H)$. □

# Main theorem

**Main Theorem (B.-)**

The Drinfeld double $D(H)$ is the center of $H$ in $\mathrm{QB}^{1-\mathrm{op}}$.

# Main theorem

$$\begin{array}{ccc} H \in \text{Hopf algebras} & \xrightarrow{\text{TK}} & \text{Monoidal categories} \ni \mathrm{LMod}(H) \\ & & \Big\downarrow \text{taking centers} \\ D(H) \in \text{Quasi-bialgebras} & \xrightarrow[\text{TK}]{} & \text{Monoidal categories} \ni \mathrm{LMod}(D(H)) \end{array}$$

# Main theorem

$$
\begin{array}{ccc}
H \in \text{Hopf algebras} & \xrightarrow{\mathrm{TK}} & \text{Monoidal categories} \ni \mathrm{LMod}(H) \\
\text{my main theorem: } \Big\downarrow \text{ taking centers} & & \text{taking center } \Big\downarrow \\
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\end{array}
$$

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**Main Theorem (B.-; in details, first part)**

The left unital action $H \to D(H) \otimes H$ is given by the quasi-bialgebra homomorphism $(\rho\colon H \to D(H) \otimes_{\Bbbk} H, t_2, t_0)$ equipped with trivial left unitality structure, where

$$\rho\colon h \mapsto \hat{1} \otimes h_{(1)} \otimes h_{(2)},$$

$$t_2 = \sum_i \hat{1} \otimes 1_H \otimes e_i \otimes e^i \otimes 1_H \otimes 1_H, \quad t_0 = 1.$$

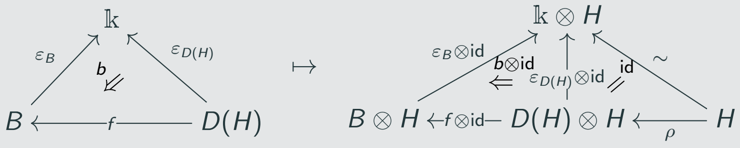
Here $\hat{1}$ is the unit of $H^*$, and $\{e_i\}_i$ is a basis of $H$ with $\{e^i\}_i$ being its dual basis.

Note that like analogous to the fact that the $E_0$-structure (monoidal functors from $\{*\}$) on a monoidal category is essentailly unique, i.e., $\mathcal{E}_0(\mathrm{MonCat}) \simeq \mathrm{MonCat}$, we have $\mathcal{E}_0(\mathrm{QB}^{1-\mathrm{op}}) \simeq \mathrm{QB}^{1-\mathrm{op}}$

**Main Theorem (B.-; in details, second part)**

The action $(\rho\colon H \to D(H) \otimes H, t_2, t_0, \mathrm{id})$ is the center of $H$ in $\mathrm{QB}^{1-\mathrm{op}}$. That is, the pushforward

$$\mathrm{QB}^{1-\mathrm{op}}(B, D(H)) \equiv \mathcal{E}_0(\mathrm{QB}^{1-\mathrm{op}})(B, D(H)) \to \mathcal{LUA}_H(B)$$




is an equivalence of categories for all quasi-bialgebras $B$.

Recall that $\mathcal{A}\mathrm{lg}_{\mathbb{k}}^{1-\mathrm{op}}$ is the 2-category of $\mathbb{k}$-algebars, reverse algebra homomorphisms and intertwinners, and $\mathrm{QB}^{1-\mathrm{op}} \equiv \mathsf{Alg}(\mathcal{A}\mathrm{lg}_{\mathbb{k}}^{1-\mathrm{op}})$.

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**Corollary**

The Drinfeld double $D(H) \in \mathsf{Alg}(\mathrm{QB}^{1-\mathrm{op}}) \equiv \mathsf{Alg}(\mathsf{Alg}(\mathcal{A}\mathrm{lg}_{\Bbbk}^{1-\mathrm{op}}))$.

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It is known that the Drinfeld double is a quasi-triangular Hopf algebra, which is hence in particular a *braided algebra* in $\mathcal{A}\mathrm{lg}_{\Bbbk}^{1-\mathrm{op}}$.

**Conjecture (Categorified Eckman-Hilton argument; Folklore)**

Let $\mathcal{C}$ be a symmetric monoidal 2-category. Then there is a canonical equivalence of 2-categories

$$\iota\colon \mathsf{Alg}(\mathsf{Alg}(\mathcal{C})) \simeq \mathrm{BrAlg}(\mathcal{C}).$$

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**Conjecture (B.-)**

The quasi-triangular structure on $D(H)$ coincides with that on $\iota(D(H))$.

**Future works and questions.**

- Verify the conjecture.
- The Meuger center of a braided monoidal category is a symmetric monoidal category. When the braided monoidal category arises from a quasi-triangular Hopf algebra $Q$, one can reconstrut a triangular Hopf algebra $F(Q)$ giving rise to the Meuger center. Can one reconstruct $F(Q)$ explicitly? Is it the center of $Q$?
- Universal property of the Drinfeld double of infinite dimensional Hopf algebras/compact quanutm groups.

**Acknowledgements.** I thank Liang Kong and Zhi-Hao Zhang for letting me know centers and the latter also for communicating me the Drinfeld center of Rep($G$).

**A related work.**

- Alain Bruguières and Alexis Virelizier. 2008. "Quantum Double of Hopf Monads and Categorical Centers." *Transactions of the American Mathematical Society*.

**Further readings.**

- Jacob Lurie. 2017. *Higher algebra*. Avaliable at: https://www.math.ias.edu/~lurie/papers/HA.pdf.
- Paddy McCrudden. 2000. "Balanced Coalgebroids". *Theory and Applications of Categories.*

Thank you for your attention! Questions and comments are welcome.